STRESS STATE OF AN ELASTICALLY SOLIDIFYING PLATE UNDER DIFFERENT HEAT-EXCHANGE CONDITIONS ON THE COOLED SURFACE

A. L. Golovanov, V. A. Zhuravlev,
A. D. Klipov, V. I. Kolodkin
and A. Yu. Stoyanov

UDC 539.319.536.5

The question of the mechanical state of media undergoing phase transformations is of great practical value. It has long been of interest and goes back to the first observations of discontinuities in the soil during its freezing and drying, to the description of fissuring of metal, glass and ceramic items during their molding, etc. The separation of the stresses into phase stresses connected with the change in density of the medium during transformation and thermal stresses due to the existence of a nonuniform temperature field and the thermal effect of the transformation was contained in the first qualitative investigations. Subsequent quantitative solutions relied upon known physical models of solidification of media. Thus, in the 1930's, Hirone [1, 2] undertook the analytical computation of the temperature stresses in a solidifying metal on the basis of a model of layer-by-layer growth of an elastic body.

In later investigations, attention was turned to the construction of physical models of the phenomenon; the description of thermoplastic flow of crystalline materials, the determination of conditions for growing dislocation-free crystals, the description of the process of crack formation and growth, the formulation of physical strength conditions for solidifying bodies, etc., were taken into account. The main contribution to the formulation and solution of many of these questions was made by Indenbom [3–8]. Methods of computing the temperature stresses in solidifying bodies were developed in parallel within the framework of the mechanics of continuous media, and the class of physical models describing the phase-transformation process was extended. A brief survey of research in this area is presented in [9]. Kosevich and Tanatarov [10] developed an appropriate analysis of phase stresses.

The one-dimensional symmetric problem of the elastic stress state of a solidifying infinite plate is examined in this paper under the assumption that the solidifying layer is prevented from bending and is free of external forces. In contrast to the preceding papers, the influence of different heat-exchange conditions on the plate surface being cooled on the distribution of the originating stresses is analyzed herein. Approximate results are obtained.

1. Let us consider a semi-infinite domain extending in the x direction and occupied by a fluid at the solidification temperature T_h. One of the heat-exchange conditions corresponding to the presence of a constant temperature or one decreasing linearly with time, decreasing exponentially, and also varying periodically, is realized on the $x=0$ surface at the initial time. The presence of a constant and a linearly time-varying heat flux and heat exchange according to Newton's law is also considered.

Let us consider the solidifying layer to satisfy the model of an elastic body, and the thermophysical and mechanical characteristics of the medium to be independent of the temperature. Such assumptions correspond to the Hirone–Indenbom–Rider model of "instantaneous solidification." To simplify the problem, let us take fixing conditions for which the growing layer is prevented from bending, which is realized, in particular, during symmetric solidification of the body or when it is grown on a rigid substrate. Using this model, let us later formulate the problem under consideration within the framework of the theory of uncoupled thermoelasticity, by taking into account that the sole nonzero components of the stress tensor which satisfy the equilibrium, and compatibility equations, and the boundary conditions are $\sigma_{yy} = \sigma_{zz} = \sigma(x, t)$ and the nonzero strain tensor components $\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon(t)$; $\varepsilon_{xx} = \varepsilon(x, t)$. We have

Gor'kii. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 162–170, January–February, 1975. Original article submitted March 6, 1974.

©1976 Plenum Publishing Corporation, 227 West 17th Street, New York, N.Y. 10011. No part of this publication may be reproduced, stored in a retrieval system, or transmitted, in any form or by any means, electronic, mechanical, photocopying, microfilming, recording or otherwise, without written permission of the publisher. A copy of this article is available from the publisher for $15.00.

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}, \quad 0 < x < \xi; \tag{1.1}$$

$$\lambda\frac{\partial T}{\partial x} = \rho Q\frac{\partial \xi}{\partial t}, \quad T(\xi, t) = T_h, \quad x = \xi; \tag{1.2}$$

$$T(x, 0) = T_h, \quad 0 \leqslant x < \infty; \tag{1.3}$$

$$T(0, t) = T_0 = \text{const} < T_h; \tag{1.4}$$

$$T(0,t) = T_0 - kt; \tag{1.5}$$

$$T(0,t) = T_0 - \Delta T \cos(wt); \tag{1.6}$$

$$\dot{T}(0, t) = T_h - \frac{Q}{C}[\exp(am^2t) - 1]; \tag{1.7}$$

$$-\lambda\frac{\partial T}{\partial x} = q_0 = \text{const}, \quad x = 0; \tag{1.8}$$

$$-\lambda\frac{\partial T}{\partial x} = q_0 - kt, \quad x = 0; \tag{1.9}$$

$$-\lambda\frac{\partial T}{\partial x} = \alpha_T[T(0, t) - T_c], \quad x = 0; \tag{1.10}$$

$$\sigma_{xx} = 0, \quad 0 \leqslant x \leqslant \xi, \quad t > 0, \tag{1.11}$$

$$\sigma(x, t) = E\int_{\tau}^{t}\{\dot{\varepsilon} - \alpha\dot{T}(x, t)\}\,dt, \quad 0 \leqslant x \leqslant \xi; \tag{1.12}$$

$$\int_0^{\xi}\dot{\sigma}(x, t)\,dx = 0, \quad 0 \leqslant x \leqslant \xi. \tag{1.13}$$

Here T is the temperature, a is the temperature conduction, λ is the heat conduction, C is the specific heat, Q is the heat of phase transition, ρ is the density, T_h is the solidification temperature, T_0 is the temperature on the surface being cooled, k, m, q_0, w are constants, α_T is the heat-exchange coefficient, t is the time, ΔT is the amplitude of the temperature fluctuations, x is the running coordinate, ξ is the coordinate of the phase boundary, τ is the time at which a fluid point is attached to the phase boundary, E is the elastic modulus, and α is the coefficient of linear expansion.

The elastic problem (1.11)-(1.13) is solved in a quasistatic formulation by using the temperature field obtained independently on the surface being cooled as the loading function in each particular case of heat exchange.

2. The approximate solution of the thermal problem corresponding to the case of a periodically varying temperature (1.1)-(1.3), (1.6), is found by following the method elucidated in [11]. Thus, the temperature field in the solidifying layer, and the law of phase boundary motion, should be found in general form from the equations

$$T_h - T = \frac{Q\rho}{\lambda}\sum_{n=0}^{\infty}\frac{1}{(2n+2)!\,a^n}\cdot\frac{d^{n+1}}{dt^{n+1}}(\xi - x)^{2n+2}, \tag{2.1}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n)!\,a^n}\frac{d^n}{dt^n}\xi^{2n} = \frac{T_h - T(0, t)}{Q}C. \tag{2.2}$$

Limiting ourselves to just the first terms of the series and taking condition (1.6) into account, we obtain

$$T_h - T = \frac{Q}{C}\left(1 - \frac{x}{\xi}\right)\frac{\xi\dot{\xi}}{a}, \qquad \xi^2 = p^2\left(1 + \delta\frac{\sin wt}{wt}\right)4at,$$

where

$$\delta = \frac{\Delta T}{T_h - T_0}; \quad 2p^2 = \frac{C(T_h - T_0)}{Q}.$$

We find the solution of the appropriate elastic problem (1.11)-(1.13) in conformity with [9], by using the temperature field presented above and the law of phase-boundary motion,

$$\frac{\sigma(p, wt_0, \theta)\,C\,(1-\nu)}{\alpha EQ} = p^2\left\{2(1 + \delta\cos wt_0\theta)\cdot\left(1 - \sqrt{\frac{\left(1 + \delta\frac{\sin wt_0\theta}{wt_0\theta}\right)\theta}{1 + \delta\frac{\sin wt_0\theta}{wt_0}}}\right) + 2\delta(\cos wt_0\theta - \cos wt_0) + \frac{1}{2}\ln\times\right.$$

$$\times\left[\frac{\left(1+\delta\frac{\sin wt_0\theta}{wt_0\theta}\right)\theta}{1+\delta\frac{\sin wt_0}{wt_0}}\right]+2\delta\int\limits_{wt_0}^{wt_0\theta}\frac{(1+\delta\cos z)\cos z}{\left(1+\delta\frac{\sin z}{z}\right)z}dz\Bigg\},\tag{2.3}$$

where $\theta = t/t_0$, $0 \leq \theta \leq 1$, t_0 is the time corresponding to the width of the solidifying part of the plate. Henceforth, σ will be understood to be the dimensionless quantity $\sigma \cdot C(1-\nu)/\alpha EQ$.

Shown in Fig. 1 is the stress distribution in the solidifying layer, computed by means of (2.3), under the following conditions:

$$p^2 = 0.1,\ \delta = 0.1,\ wt_0 = \ 1)\ \frac{\pi}{6},\ 2)\ \pi,\ 3)\ \frac{3}{2}\pi,\ 4)\ 2\pi.$$

It is easy to see that the stress distribution originating is sign-varying. The stresses on the surface being cooled are hence always negative and tend to infinity. The plastic compression domain is evident here.

If it is considered that $\delta = 0$ ($\Delta T = 0$), then the plate surface being cooled is maintained at the constant temperature $T = T_0 < T_h$. Under these conditions, the law of phase-boundary motion reduces to a simplified modification of the classical solution of the Stefan problem (1.1)-(1.4),

$$\xi^2 = \frac{2\lambda (T_h - T_0)}{\rho Q} t.$$

At the same time, the general solution (2.3) takes a form agreeing with the solution presented in [9] for $p \ll 1$

$$\sigma\left(p, \frac{x}{\xi}\right) = p^2\left(\ln\frac{x}{\xi} - 2\frac{x}{\xi} + 2\right).\tag{2.4}$$

Curve 1 in Fig. 2 corresponds to this stress distribution under the conditions $p^2 = 0, 1$. The coordinate of the maximal tensile stresses in the solidifying layer x_m follows from the condition $\partial_x \sigma = 0$ and equals $x_m/\xi \approx 0.5$ in the case (2.4).

3. In the case of a linearly decreasing temperature, the approximate solution of the thermal problem (1.1)-(1.3), (1.5) is found from the general solutions (2.1), (2.2) under constraints analogous to those taken in Sec. 2,

$$T_h - T = 2p^2\frac{Q}{C}\left(1 - \frac{x}{\xi}\right)\sqrt{1 - m^2\xi^2},\tag{3.1}$$

$$\xi^2 = p^2 \cdot 4at + 2p^2 m^2 a^2 t^2,\tag{3.2}$$

where $m^2 = \frac{kC}{2p^2 aQ}$, $\quad 2p^2 = \frac{T_h - T_0}{Q} C$.

Let us also note that analogous heat-exchange conditions are also realized for the periodic cooling mode in the plate solidification stage when $wt_0 \approx \pi/2(2n-1)$, $n = 1, 2, \ldots$. Using (3.1), (3.2), let us write the solution of the corresponding elastic problem,

$$\sigma\left(p, m\xi, \frac{x}{\xi}\right) = p^2\left[2\left(\sqrt{1 - m^2\xi^2}\right)\left(1 - \frac{x}{\xi}\right) - 2\left(\sqrt{1 - m^2\xi^2} - \right.\right.$$

$$\left.\left. - \sqrt{1 - m^2\xi^2\frac{x^2}{\xi^2}}\right) - \frac{1}{2}\ln\frac{\left(1 - \sqrt{1 - m^2\xi^2}\right)\left(1 + \sqrt{1 - m^2\xi^2\frac{x^2}{\xi^2}}\right)}{\left(1 + \sqrt{1 - m^2\xi^2}\right)\left(1 - \sqrt{1 - m^2\xi^2\frac{x^2}{\xi^2}}\right)}\right].$$

For $m^2\xi^2 \ll 1$ this solution can be simplified:

$$\sigma\left(p, m\xi, \frac{x}{\xi}\right) = p^2\left[\ln\frac{x}{\xi} - 2\frac{x}{\xi} + 2 + m^2\xi^2\frac{x}{\xi}\left(1 - \frac{x}{\xi}\right) - \frac{1}{2}\ln\frac{4 - m^2\xi^2\frac{x^2}{\xi^2}}{4 - m^2\xi^2}\right].\tag{3.3}$$

If $m = 0$, then (3.3) agrees with (2.4). The coordinate of the maximal tensile stresses for the simplified distribution (3.3) follows from

$$\frac{\xi}{x_m} + m^2\xi^2\left(1 - 2\frac{x_m}{\xi}\right) + \frac{m^2\xi^2\frac{x_m}{\xi}}{4 - m^2\xi^2\frac{x_m^2}{\xi^2}} = 2.$$

The stress distribution (3.3) is presented in Fig. 2 (curve 2) for $p^2 = 0.1$; $m\xi = 0.705$.

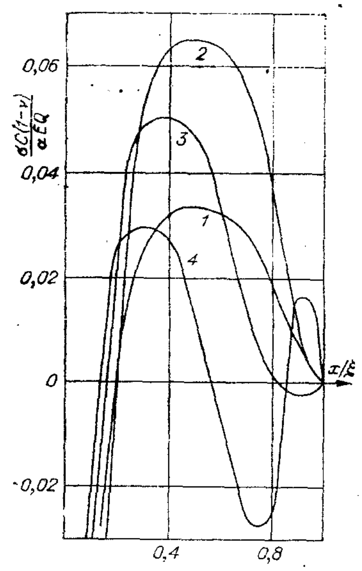

Fig. 1

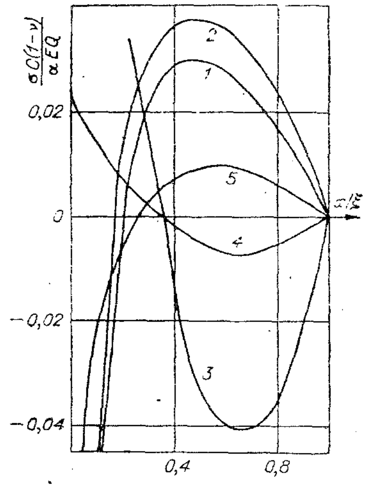

Fig. 2

4. The exact solution of the thermal problem in the case of an exponentially decreasing temperature (1.1)-(1.3), (1.7) is according to [12]

$$T_h - T = \frac{Q}{C}\left\{\exp\left[m\xi\left(1 - \frac{x}{\xi}\right)\right] - 1\right\},$$
$$\xi = mat, \quad m = \text{const.}$$

The solution of the corresponding elastic problem results in the following thermal stress distribution in the solidifying layer:

$$\sigma\left(m\xi, \frac{x}{\xi}\right) = \exp\left[m\xi\left(1 - \frac{x}{\xi}\right)\right] - 1 - \ln\frac{x}{\xi} + \mathrm{E}_i\left(m\xi\frac{x}{\xi}\right) - \mathrm{E}_i(m\xi), \tag{4.1}$$

where $\mathrm{E}_i(y) = \int\limits_{-\infty}^{y} \frac{\exp(y)}{y}\,dy$.

We have on the plate surface being cooled

$$\sigma(m\xi,\ 0) = \exp(m\xi) - 1 + C_1 + \ln(m\xi) - \mathrm{E}_i(m\xi),$$

where $C_1 = 0.577$ is the Euler constant.

For small values of the parameter $m\xi \ll 1$ we have $\sigma(m\xi,\ 0) \to 1/4(m\xi)^2$, and, therefore, the greatest tensile stresses are found on the plate surface being cooled.

5. The approximate solution of the problem corresponding to the case of a constant heat flux (1.1)-(1.3), (1.8) can be found by again returning to the method elucidated in [11]. The temperature field is hence represented by (2.1), and the law of phase-boundary motion is found from the equation

$$\sum_{n=0}^{\infty} \frac{1}{a^n (2n+1)!} \frac{d^{n+1}}{dt^{n+1}} \xi^{2n+1} = ma. \tag{5.1}$$

Limiting ourselves to two terms of the series in (2.1), and to one in (5.1), we obtain

$$T_h - T = \frac{Q}{C}\left[\left(1 - \frac{x}{\xi}\right)\frac{\xi\dot{\xi}}{a} + \left(1 - \frac{x}{\xi}\right)^2 \frac{\xi^2\dot{\xi}^2}{2a^2}\right],$$
$$\xi = mat, \quad m = \frac{q_0 C}{\lambda Q}.$$

The simplified temperature field, used in the preceding section when $m\xi[1-(x/\xi)] \ll 1$, yields these solutions of the thermal problem.

The solution of the elastic problem for this case is

$$\sigma\left(m\xi, \frac{x}{\xi}\right) = \frac{m^2\xi^2}{4}\left(1 - 4\frac{x}{\xi} + 3\frac{x^2}{\xi^2}\right). \tag{5.2}$$

The greatest tensile stresses $\sigma(m\xi, 0) = 1/4(m\xi)^2$ originate on the plate surface being cooled ($x = 0$). The coordinate of the maximum compressive stresses corresponds to the quantity $x_m/\xi = 2/3$.

The stress distribution found by means of (5.2) for $m\xi = 0.705$ is shown in Fig. 2 (curve 3).

6. We find the approximate solution of the thermal problem in the case of a linearly decreasing heat flux (1.1)-(1.3), (1.9) by using (2.1) and (5.1) analogously to the solution examined in the previous section. We have

$$T_h - T = \frac{Q}{C}\left[\left(1 - \frac{x}{\xi}\right)\frac{\xi\dot{\xi}}{a} + \frac{1}{2}\left(1 - \frac{x}{\xi}\right)^2 \frac{\xi^2\dot{\xi}^2}{a^2} + \frac{1}{6}\left(1 - \frac{x}{\xi}\right)^3 \frac{\xi^3\ddot{\xi}}{a^2}\right],$$
$$\xi = mat\left(1 - \frac{kt}{2q_0}\right), \quad m = \frac{q_0 C}{\lambda Q}.$$

The solution of the elastic problem yields the thermal stress distribution in the solidifying layer,

$$\sigma\left(m\xi, \frac{x}{\xi}\right) = -nm\xi\,\frac{x}{\xi}\left(1 - \sqrt{\frac{x}{\xi}}\right)\sqrt{1 - nm\xi} + \frac{m^2\xi^2}{4}\left[1 - \right.$$
$$\left. -4(1 - nm\xi)\frac{x}{\xi} + \frac{3x^2}{\xi^2} - 2nm\xi\left(\frac{4}{9} + \frac{x^2}{\xi^2} + \frac{5}{9}\frac{x^3}{\xi^3}\right)\right], \tag{6.1}$$

where $n = \dfrac{2kQ^2\rho^2 a}{q_0^3}$.

The maximum tensile stresses

$$\sigma(m\xi, 0) = \frac{m^2\xi^2}{4}\left(1 - \frac{8}{9}nm\xi\right).$$

originate on the plate surface being cooled.

In the limit case $n = 0$ ($k = 0$) the solution (6.1) agrees with the solution (5.2). The stress distribution computed by means of (6.1) for $m\xi = 0.333$ at $n = 0.1$ is shown in Fig. 2 (curve 4).

7. In the case of heat exchange according to a Newton law, we find the temperature field and law of phase-boundary motion by solving the thermal problem (1.1)-(1.3), (1.10). To do this, we again use the results from [12]. The temperature field is given by (2.1), and we have the following equation for the law of phase-boundary motion:

$$\sum_{n=1}^{\infty} \frac{1}{a^n}\frac{d^n}{dt^n}\left[\frac{\xi^{2n}}{(2n)!} + \frac{\lambda}{\alpha_T}\frac{\xi^{2n-1}}{(2n-1)!}\right] = \frac{T_h - T_c}{Q}C. \tag{7.1}$$

Limiting ourselves to the first terms of the expansions (2.1) and (7.1), we obtain

$$T_h - T = \frac{Q}{C}\left(1 - \frac{x}{\xi}\right)\frac{\xi\dot{\xi}}{a},$$
$$\xi^2 = 4p^2 at - \frac{2}{m^2}\left(\sqrt{1 + 4p^2 atm^2} - 1\right),$$
$$2p^2 = \frac{T_h - T_c}{Q}C, \quad m = \frac{\alpha_T}{\lambda}.$$

The solution of the elastic problem for this case reduces to the following temperature stress distribution in the solidifying layer:

$$\sigma\left(m\xi, \frac{x}{\xi}\right) = p^2\left[\frac{2m\xi}{1 + m\xi}\left(1 - \frac{x}{\xi}\right) + \ln\frac{1 + m\xi\frac{x}{\xi}}{1 + m\xi} + \left(\frac{1}{m\xi + 1} - \frac{1}{1 + m\xi\frac{x}{\xi}}\right)\right]. \tag{7.2}$$

It follows from (7.2) that compressive stresses whose magnitude equals

$$\sigma(p, m\xi, 0) = p^2\left[\frac{m\xi}{1 + m\xi} - \ln(1 + m\xi)\right]$$

originated on the plate surface being cooled, and the value of the maximal tensile stress coordinate in the solidifying layer is determined by the relationship

$$\frac{x_m}{\xi} = \frac{1}{4}\left(1 + \sqrt{\frac{9 + m\xi}{1 + m\xi}}\right) - \frac{1}{m\xi}\left[1 - \frac{1}{4}\left(1 + \sqrt{\frac{9 + m\xi}{1 + m\xi}}\right)\right].$$

The stress distribution computed by means of (7.2) for $p^2 = 1.0$ and $m\xi = 0.333$ is presented in Fig. 2 (curve 5).

8. It follows from the results presented that the heat-exchange conditions substantially affect the nature of the temperature distribution in the solidifying plate and the associated law of phase-boundary motion. This has a number of consequences; primarily, the presence of qualitative changes in the originating thermal stress distribution. In this sense, the heat-exchange conditions considered can be separated into two groups according to a criterion governing qualitatively identical pictures of the stress state of the growing plate.

Among the first should be the case of heat exchange according to the laws (1.4)-(1.6), (1.10). The effective stress distribtuion under these conditions is characterized by the presence of a compression domain at the surface being cooled, where $\sigma|_{x=0} \to -\infty$. Here the most general case is a periodic temperature mode at the plate surface being cooled, which corresponds to (1.6), in which the heat-exchange conditions with a constant temperature and temperatures decreasing and increasing linearly according to a parabolic law are contained as particular cases. In this connection, let us examine some singularities in the stress distribution which are characteristic for the periodic cooling mode. Thus, the stress state of the plate at the start of the period of temperature variation is characterized by two zones (see Fig. 1, curve 1): one growing at the phase boundary and a compression zone at the surface being cooled. No qualitative changes in the stress distribution (curve 2) originate during a time equal to half the period of temperature variation, as compared with the beginning of the period although the maximal tensile stresses grow. A stress redistribution with the formation of a small compression domain at the phase boundary (curve 3) occurs in the next quarter of the period. Finally, the picture changes abruptly during the last quarter, the compression zone originating broadens and shifts to the middle layer of the solidifying part of the plate, and a narrow, growing domain with stresses of considerable magnitude appears near the phase boundary. As a whole, alternation of elastic compressive and tensile zones whose width and maximal effective stresses depend on the magnitude of the parameter δ, is characteristic of the stress state at this time. For small δ the influence of the temperature fluctuations on the stress distribution is negligible. As $\delta > 1$ grows, the originating stresses turn out to be proportional to δ. No qualitative changes in the stress state of the growing plate originate for these changes in the quantity δ. Variation of the other external parameter, the frequency of the temperature fluctuations, results in the following singularities: the stress distribution tends to the expression (2.4) in the case $w \to \infty$ and the number of compressive and tensile zones in the solidifying layer grow correspondingly for $w \to \infty$.

The case of heat exchange according to a Newton law which abuts the first group is characteristic in that the stress distribution (7.2) tends to the distribution (2.4) as the heat-transport coefficient grows ($m \to \infty$, i.e., when ideal contact conditions between the surface being cooled and the surrounding medium hold). On the other hand, no stresses originate for $m = 0$.

Realization of the heat-exchange conditions corresponding to the laws (1.7)-(1.9) yields another qualitatively coincident picture of the stress distribution in the solidifying plate. Here the tensile stress zone adjoins the cooled surface, and a compressive domain originates at the phase boundary. It is characteristic that the tensile stresses on the plate surface being cooled are finite. The most general case is hence that with the exponentially decreasing temperature (4.1).

LITERATURE CITED

1. T. Hirone, "Internal stress due to cooling in spherical steel ingots," Science Reports of Tohoku Imperial University. Professor Honda's Anniversary Volume, 1017 (1936).
2. T. Hirone, "Internal stress due to cooling in cylindrical steel ingots," Science Reports of Tohoku Imperial University, Ser. 1, 26, 214 (1937).
3. V. L. Indenbom, "On the theory of hardening glass," Zh. Tekh. Fiz., 24, No. 5, 925 (1954).
4. V. L. Indenbom, "On stresses on a glass article surface," Zh. Tekh. Fiz., 26, No. 2, 370 (1956).
5. V. L. Indenbom, "Macroscopic theory of dislocation formation during crystal growth," Kristallografiya, 2, No. 5, 594 (1957).
6. V. L. Indenbom, "On the theory of stress and dislocation formation during crystal growth," Kristallografiya, 9, No. 1, 74 (1964).
7. V. L. Indenbom and L. I. Vidro, "Thermoplastic and structural stresses in solids," Fiz. Tverd. Tela, 6, No. 4, 992 (1964).
8. V. L. Indenbom, I. S. Zhitomirskii, and T. S. Chebanova, "Theoretical investigation of the stresses originating during crystal growth," Growth of Crystals [in Russian], Vol. 8, Nauka, Moscow (1968), p. 303.

9. V. A. Zhuravlev and V. R. Fidel'man, "Temperature stresses in a continuous flat ingot," Inzh.-Fiz. Zh., 23, No. 3, 519 (1972).
10. A. M. Kosevich and L. V. Tanatarov, "Plastic deformation and irreversible changes in a solid during local melting. Filamentary heat source," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 5, 61 (1961).
11. V. T. Borisov, B. Ya. Lyubov, and D. E. Temkin, "On the computation of the kinetics of metal ingot solidification for different temperature conditions on its surface," Dokl. Akad. Nauk SSSR, 104, No. 2, 223 (1955).
12. H. S. Carslaw and J. C. Jaeger, Conduction of Heat in Solids, Oxford University Press (1959).